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Physical Quantities in a Classical Two-Tensor Theory of Gravitation

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Abstract

We compute the energy content inside a three-dimensional sphere surrounding a point mass, which is taken to be the source of the short-range tensor field coupled to the long-range gravitational field through the equations of motion of the *f–g* theory.

1. *Introduction*

Several exact solutions have been given (Aichelburg, 1973) to the 1970 *f–g* theory (Isham, Abdus Salam & Strathdee, 1971). Recently, this work has been generalised to include internal symmetries (Isham, Abdus Salam & Strathdee, 1974). In the present article we shall restrict ourselves to the original coupling of the strong and weak gravity tensors.

However, although exact solutions are known, none of them is spherically symmetric, which in Einstein's General Theory of Relativity has proved to be of great value for testing the theory.

On the other hand, at the classical level it is useful to derive exact, or nearly exact, results to see in which ways does *f–g* differ from Einstein's 1916 theory.

In Aragone & Chela-Flores (1972) an asymptotic spherically symmetric solution to the 1970 theory was found. In the present work we reconsider that solution; the purpose of looking further at that solution is that in order to calculate physically relevant quantities (energy, total momentum, angular momentum, etc.) of an arbitrary finite system, it is only necessary to know the asymptotic behaviour of the field at great distances; thus, in the present work, we choose to evaluate one of the quantities enumerated above (energy) in spite of the fact that an exact solution (of the Schwarzschild type) remains unknown. Therefore, we discuss the problem of evaluating the energy of the *f*-field (against a Minkowskian background); this was essentially the physical

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situation for which we solved the f-field equations (in the limit $\kappa_g \to 0$) obtaining the expected Yukawa behaviour (Aichelburg, 1973; Aragone & Chela-Flores, 1972).

We take the simple case when the source of the f-field is a point mass M. We enclose it in a sphere of large radius and then move out the boundary surface to a distance where the field is practically linear. Since we possess a solution of the field at this distance, we can obtain the energy by evaluating a certain surface integral surrounding the particle, without having to penetrate the strongly non-linear inner core of the f-field; in other words, without having found an *exact* spherically symmetric solution.

The plan of this paper is as follows:

In Section 2 we sketch the method for obtaining the asymptotic solution of Aragone & Chela-Flores (1972), which was not published in detail (and relegate to the Appendix all the tedious details).

In Section 3 we present the derivation of an integral expression for the energy-momentum four vector P_μ from which the formula for the total energy of the f-field inside the large sphere is inferred.

In Section 4 we write the line element in isotropic coordinates.

Finally, in Section 5, we fix the undetermined constant of the asymptotic solution by looking at the classical limit of f-gravity, when the potential is taken to be Yukawa's; we end this section by giving the expression for the total energy.

2. *The Asymptotic Spherically Symmetric Solution*

In order to establish the nomenclature let us recall that in Aragone & Chela-Flores (1972) we looked for a Schwarzschild type of solution for the f- field; for that purpose we expected the theory to have, for long-range gravitation, all the good features of Einstein's theory; in particular we single out:

(a) In the absence of gravitation we ought to recover special relativity; this is expressed by saying that asymptotically

$$g^{\alpha\beta} = g^{(M)\alpha\beta}; \qquad g^{(M)\alpha\beta} = dg(1, -1, -1, -1) \tag{2.1}$$

where dg is the diagonal matrix.

Further, suppose that we have a hadronic test particle (hadronic in the sense that is is the source of the f-field); then, from the relative strengths of κ_g and κ_f we expect the effect of g-gravity to be negligible with respect to f-gravity. Hence, we are led to look at the limit of the theory where $\kappa_g \equiv 0$. Then we are only left with the f-field equations.

In order to solve that set we introduce spherical symmetry and let

$$f^{\alpha\beta} = dg(D(r), -A(r), -B(r)r^2, -B(r)r^2 \sin^2 \theta) \tag{2.2}$$

We remark, that following Isham, Abdus Salam & Strathdee (1971), we define the diagonal elements of the f-field with *contravariant* indices, rather than the more usual convention of Adler, Bazin & Schiffer (1965), care has been needed in what follows, particularly starting from equation (4.10).

We then search for a solution for the f-tensor. Let

$$D(r) = 1 - u(r) \tag{2.3a}$$

$$B(r) = 1 - v(r) \tag{2.3b}$$

$$A(r) = 1 - w(r). \tag{2.3c}$$

Another feature of strong gravity is clear: the new (hadronic) potential shall be of short range (this is verified *a posteriori* by our asymptotic solution); thus, we expect, *a priori*, some sort of exponentially decreasing behaviour for the functions u, v and w; therefore, this leads us to a second important property.

(b) Second-order terms in the functions u, v and w are negligible; in other words, we are led to linear field equations for the f-$g^{(M)}$ theory.

The physically simplest case to deal with (and the analogous one to the Schwarzschild exterior solution) is the vacuum field equations.

In this case the f-field equations are written, following Isham, Abdus Salam & Strathee (1971), as

$$\frac{G_{\mu\nu}(f)}{k_f^2\sqrt{(-f)}} + \frac{\partial \mathscr{L}_{fg}}{\partial f^{\mu\nu}} = 0 \tag{2.4a}$$

which may be rewritten (cancelling some factors) as in Aragone & Chela-Flores (1972):

$$\frac{1}{m^2} G^{\mu}{}_{\nu}(f) = -S^{\mu}{}_{\nu} \tag{2.4b}$$

We notice that with the convention underneath equations (2.2), the determinant f is defined as $f \equiv \det f_{\alpha\beta}$.

By explicit evaluation of the algebraic terms $S^{\mu}{}_{\nu}$ we find

$$S^1{}_1 = \tfrac{1}{4}(-2u - 4v + 2wv + wu - v^2 - 2uv)$$
$$S^2{}_2 = \tfrac{1}{4}(-2u - 2v - 2w - v^2 + uw)$$
$$S^4{}_4 = \tfrac{1}{4}(-2w - 4v - v^2 + uw + 2uv - 2vw)$$

But from property (b) these terms may be simplified to

$$S^1{}_1 = -\tfrac{1}{2}(u + 2v) \tag{2.5a}$$

$$S^2{}_2 = -\tfrac{1}{2}(u + v + w) \tag{2.5b}$$

$$S^3{}_3 = -\tfrac{1}{2}(w + 2v) \tag{2.5c}$$

It is convenient to write

$$x = rm_f, \qquad m_f \text{ mass of } f\text{-meson}$$

Then, in view of property (b) the various components of the Einstein tensor are

$$G^1{}_1 = \frac{v - w}{x^2} + \frac{v'}{x} + \frac{u'}{x} \tag{2.6a}$$

$$G^2{}_2 = \tfrac{1}{2}(v'' + u'') + \frac{1}{x} \quad - \frac{w'}{2} + v' + \tfrac{1}{2}u' \tag{2.6b}$$

$$G^4{}_4 = \frac{v - w}{x^2} + \frac{3v'}{x} - \frac{w'}{x} + v'' \tag{2.6c}$$

Then gathering together equations (2.5) and (2.6), through the field equations (2.4b), we may write

$$\frac{v - w}{x^2} + \frac{v'}{x} + \frac{u'}{x} = \tfrac{1}{2}(u + 2v) \tag{2.7a}$$

$$\tfrac{1}{2}(v'' + u'') + \frac{v'}{x} + \frac{u'}{2x} - \frac{w'}{2x} = \tfrac{1}{2}(u + v + w) \tag{2.7b}$$

$$\frac{v - w}{x^2} + \frac{3v'}{x} - \frac{w'}{x} + v'' = \tfrac{1}{2}(w + 2v) \tag{2.7c}$$

The solution of this system of equations is straightforward, though lengthy. In the Appendix we give the necessary details and infer, in equations (A.17), (A.18) and (A.19), that the complete solution of our f-$g^{(M)}$ system, linearised and with $\kappa_g = 0$ is

$$u = a \exp(-x)/x \tag{2.8a}$$

$$v = -a \exp(-x)/2x \tag{2.8b}$$

$$w = a \exp(-x)/x^2 \tag{2.8c}$$

where the leading terms are retained (though some first-order corrections were given in (A.18) and (A.19).

3. *Integral Expressions for the Four-Vector P_μ*

We begin this section by recalling that if f is the metric tensor one may write the Einstein tensor as

$$G_{\mu\nu}(f) = \frac{\partial \mathscr{L}_f}{\partial f^{\mu\nu}} - \partial_\lambda \frac{\partial \mathscr{L}_f}{\partial \partial_\lambda f^{\mu\nu}} \tag{3.1}$$

where $\mathscr{L}_f$ is the Einstein lagrangian density

$$\mathscr{L}_f = f^{\sigma\rho} \left[\begin{Bmatrix} \alpha \\ \sigma\ \rho \end{Bmatrix} \begin{Bmatrix} \beta \\ \alpha\ \beta \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \sigma\ \rho \end{Bmatrix} \begin{Bmatrix} \beta \\ \alpha\ \sigma \end{Bmatrix} \right] \tag{3.2}$$

Then, if we write the mixing (interaction) term as

$$\mathscr{L}_{fg} = \frac{m^2}{4} F^{\alpha\beta} F^{\kappa\lambda} (g_{\alpha\kappa} g_{\beta\lambda} - g_{\alpha\beta} g_{\kappa\lambda}) \frac{1}{\kappa_f^2 \sqrt{(-f)}} = \frac{1}{\kappa_f^2 \sqrt{(-f)}} L_{fg};$$

with $F^{\alpha\beta} = f^{\alpha\beta} - g^{\alpha\beta}$ one is able to write the f-tensor field equations as

$$G_{\mu\nu}(f) + \frac{\partial L_{fg}}{\partial f^{\mu\nu}} = 0 \tag{3.3}$$

Then introducing the Einstein Lagrangian density (cf. equations (3.1) and (3.2)) we may rewrite equations (3.3) as

$$\frac{\partial \mathscr{L}_f}{\partial f^{\mu\nu}} - \partial_\lambda \frac{\partial \mathscr{L}_f}{\partial \partial_\lambda f^{\mu\nu}} + \frac{\partial L_{fg}}{\partial f^{\mu\nu}} = 0$$

or equivalently

$$\frac{\partial \mathscr{L}_{\mathrm{TOT}}}{\partial f^{\mu\nu}} - \partial_\lambda \frac{\partial \mathscr{L}_{\mathrm{TOT}}}{\partial \partial_\lambda f^{\mu\nu}} = 0 \tag{3.4}$$

having noticed, in writing equation (3.4), that the total Lagrangian given by

$$\mathscr{L}_{\mathrm{TOT}} = \mathscr{L}_g + \mathscr{L}_f + L_{fg}$$

has no f dependence in $\mathscr{L}_g$ and that $\mathscr{L}_{\mathrm{TOT}}$ has no first derivatives of the f-field in the "algebraic" mixing term L_{fg}.

We are therefore able to write the four-vector P_μ as

$$P_\mu = \frac{c^2}{8\pi\kappa_f^2} \int_{V^3} \partial_\lambda \left(\frac{\partial \mathscr{L}_{\mathrm{TOT}}}{\partial \partial_\lambda f^{\mu\nu}} f^{0\nu} \right) d^3x \tag{3.5}$$

where V^3 is the entire space at a given time (cf. Adler, Bazin & Schiffer (1965), p. 320). Further, as $\mathscr{L}_g$ and L_{fg} do not depend on the field derivatives of the f-field, one can equally well write for the zeroth component of P_μ,

$$P_0 = \frac{c^2}{8\pi\kappa_f^2} \int_{V^3} \partial_\lambda \left(\frac{\partial \mathscr{L}_f}{\partial \partial_\lambda f^{0\nu}} f^{0\nu} \right) d^3x$$

Finally, for a spherically symmetric f-field $f^{\mu\nu} = 0$ if $\mu \neq \nu$, hence

$$P_0 = \frac{c^2}{8\pi\kappa_f^2} \int_{V^3} \partial_\lambda \left(\frac{\partial \mathscr{L}_f}{\partial \partial_\lambda f^{00}} f^{00} \right) d^3x \tag{3.6}$$

However, in writing formula (3.5), an important point should be emphasised: we are able to identify P_μ as a *conserved quantity* provided that, at infinity, the integrand of (3.5) behaves appropriately, namely, tends to zero. This will not be the case with our spherically symmetric solution (2.8), since we recall that from equations (2.2) and (2.3) there is still an r^2 dependence, which when put into the Einstein Lagrangian and differentiated *once* will not be null. This difficulty is overcome in the Einstein theory by transforming the spherically symmetric solution into isotropic coordinate. This will be done, in our case, in the next section.

4. *Isotropic Coordinates*

As in the usual method, we choose to write the line element in its isotropic form

$$ds^2 = (1 - a \exp(-x)/x)c^2\, dt^2 - \lambda^2(\rho)\, d\sigma^2 \tag{4.1}$$

where $d\sigma^2$ is $dx^2 + dy^2 + dz^2$ in cartesian coordinates, $dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi$ in spherical coordinates, etc. We search for two functions $r(\rho)$ and $\lambda^2(\rho)$ for which (4.1) shall coincide with

$$\begin{aligned} ds^2 = (1 - a \exp(-x)/x)c^2\, dt^2 - (1 - a \exp(-x)/x^2)\, dr^2 \\ - (1 + a \exp(-x)/2x) r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \end{aligned} \tag{4.2}$$

Comparing coefficients of $(d\theta^2 + \sin^2\theta\, d\phi^2)$ in (4.1) and (4.2) we find

$$(1 + a \exp(-x)/2x) r^2 = \lambda^2 \rho^2 \tag{4.3}$$

On the other hand, comparing coefficients of radial intervals, we find

$$(1 - a \exp(-x)/x^2)\, dr^2 = \lambda^2\, d\rho^2 \tag{4.4}$$

Substituting λ from (4.3) into (4.4) yields

$$(1 - a \exp(-x)/x^2)\, dx^2 = (1 + a \exp(-x)/2x) x^2 (d\rho/\rho)^2$$

Then, to lowest order,

$$\pm (1 - a \exp(-x)/4x)\, dx/x = d\bar{\rho}/\bar{\rho} \tag{4.5}$$

where we have introduced the convenient notation $\bar{\rho} = \rho m_f$. We shall integrate equation (4.5) retaining the positive sign (this choice becomes clear below for physical reasons). Let, for simplicity $a/4 = A$. Then, integrating equation (4.5), we find

$$\log x - A \int [\exp(-x)/x^2]\, dx = \log \bar{\rho} \tag{4.6}$$

In order to perform the above integral we introduce the following convenient notation:

$$I_0 = \int [\exp(-x)/x^2]\, dx$$

$$I_1 = \int [\exp(-x)/x]\, dx$$

$$I_2 = \int \log x \exp(\lambda x)\, dx, \qquad \lambda = -1$$

By successive partial integrations we find

$$I_0 = -\exp(-x)/x - I_1 \tag{4.7a}$$

$$I_1 = \exp(-x)\log x + I_2 \tag{4.7b}$$

Then from Gröbner and Hofreiter (1961) we find

$$I_2 = -\exp(-x)\log x - E_1(x) + C \tag{4.7c}$$

where $E_1(x) = -\mathcal{E}_i(-x)$ and $\mathcal{E}_i(x)$ is the exponential integral

$$\mathcal{E}_i(x) = \int_{-\infty}^{x} [\exp(t)/t]\, dt$$

Next, from (4.7b) we find

$$I_1 = \exp(-x)\log x + (-\exp(-x)\log x - E_1(x)) + C$$

Hence

$$I_1 = -E_1(x) + C$$

Therefore, in (4.7a) we find

$$I_0 = -\exp(-x)/x + E_1(x) + C$$

which implies, through equation (4.6), that

$$\log x - AI_0 = \log \bar{\rho} + C$$

or, equivalently,

$$\log x + a\exp(-x)/4x - aE_1(x)/4 = \log(\bar{\rho}/\bar{\rho}_0) \tag{4.8}$$

having included the arbitrary constant of integration in the logarithmic function on the right-hand side of (4.8). As $x \to \infty$, $E_1(x) \to 0$ (Abramowitz & Stegun, 1965). Thus, for x large enough,

$$\log(\bar{\rho}/\bar{\rho}_0) \sim \log x$$

Therefore, with $\bar{\rho}_0 = 1$, we manage to have both r and ρ behaving similarly for large radial distances, thus justifying the + sign choice in equation (4.5).

We find from equation (4.3) that

$$\lambda^2 = 1 + a \exp(-\bar{\rho})/2\bar{\rho}$$

Hence, asymptotically, we may write the line element (4.2) as

$$\begin{aligned} ds^2 &= (1 - a \exp(-\bar{\rho})/\bar{\rho})c^2\, dt^2 \\ &\quad - (1 + a \exp(-\bar{\rho})/2\bar{\rho})\,[d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \end{aligned} \tag{4.9}$$

Computing $f^{00}\,\partial\mathscr{L}_f/\partial\partial_j f^{00}$ we find (cf. Adler, Bazin & Schiffer (1965), p. 323)

$$f^{00}\,\frac{\partial\mathscr{L}_f}{\partial\partial_j f^{00}} = -\sqrt{(-\det f^{\alpha\beta})}\,\partial_j f^{11} \tag{4.10}$$

In order to evaluate the integrand of P_0 as given in (4.10), we first evaluate

$$\begin{aligned} \sqrt{(-\det f^{\alpha\beta})} &= (1 - a \exp(-\bar{\rho})/\bar{\rho})^{1/2}(1 + a \exp(-\bar{\rho})/2\bar{\rho})^{3/2} \\ &\cong 1 + a \exp(-\bar{\rho})/4\bar{\rho} \end{aligned}$$

We have emphasized, in the introduction, that for P_μ we have the curious result that the generalised energy-momentum P_μ of a volume V^3 may be determined from the values of the metric tensor field and its derivatives on the surface of V^3; the detailed behaviour of the field inside V^3 is irrelevant.

In order to exploit this feature we return to (3.6) and use Gauss' theorem,

$$P_0 = \frac{c^2}{8\pi\kappa_f^2}\int \frac{\partial\mathscr{L}_f}{\partial\partial_j f^{00}}\, f^{00} n_j\, dS$$

Hence with equation (4.10)

$$P_0 = -\frac{c^2}{8\pi\kappa_f^2}\int \sqrt{(-\det f^{\alpha\beta})}\,\mathrm{grad}\, f^{11}.\, dS \tag{4.11}$$

Therefore, in this theory we find

$$\partial_\rho f^{11} = \partial_\rho(1 + a \exp(-\bar{\rho})/\bar{\rho}) \cong -a \exp(-\bar{\rho})/\rho$$

Finally, from equation (4.11) we find

$$P_0 \cong \frac{Rc^2}{2\kappa_f^2}\, a \exp(-m_f R) \tag{4.12}$$

where R is the radius of the sphere encircling the hadronic point source.

5. *The Classical Yukawa Limit*

In order to determine the arbitrary constant of integration a in the final form of the energy content P_0, we shall study the classical limit of strong gravity.

For that purpose we shall apply the geodesic equations of motion to a Riemann space with an f-metric and shall also suppose that the velocity of the test particle along the geodesic is much less than c. Let us write

$$f^{\alpha\beta} = f^{(M)\alpha\beta} + F^{\alpha\beta}; \qquad f^{(M)} = dg(1, -1, -1, -1)$$

Our classical limit is obtained in the same way as the classical limit of general relativity and we find (cf. Adler, Bazin & Schiffer (1965), p. 124),

$$\frac{d^2}{dt^2}\mathbf{x} = -\frac{c^2}{2}\,\partial_{\mathbf{x}} F^{00}$$

which may be taken to be Newton's equation of motion in a Yukawa field derived from a scalar potential, provided we identify the scalar potential as

$$\phi = \frac{c^2}{2} F^{00}$$

Therefore, geodesic motion will occur in a classical potential ϕ if

$$F^{00} = 2\phi/c^2 \tag{5.1}$$

Comparing our solution (2.8) with (5.1), we find

$$f^{00} \cong 1 - 2\kappa_f^2 M \exp(-m_f r)/rc^2 \tag{5.2}$$

having chosen a Yukawa-type potential, instead of the usual Newtonian potential,

$$V = -\kappa_f^2 M \exp(-m_f r)/r$$

where M is the source of the f-field; the damping factor $\exp(-m_f r)$ has been included in order to give the classical limit it required short-range character. However, (2.8) and (5.2) yield

$$a \exp(-m_f r)/rm_f = 2\kappa_f^2 M \exp(-m_f r)/rc^2$$

Hence, we conclude that

$$a = 2\kappa_f^2 M m_f/c^2 \tag{5.3}$$

Therefore the formula for the total *energy* is given by equations (4.12) and (5.3), as $c^2 P_0$ where,

$$P_0 = M m_f R \exp(-m_f R) \tag{5.4}$$

Finally, as we might expect, for $R \sim 10^{-13}$ cm (nuclear range), $P_0 \sim M$ (since $m_f \sim 10^3$ MeV), not in disagreement with special relativity.

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Appendix

In order to find a solution to the system of equations (2.7) we find it is simpler to solve the following system.

Instead of equation (2.7b) we consider (2.7c) minus twice (2.7b),

$$x^2u'' - xv' + xu' + w - v = \frac{x^2}{2}(2u + w) \tag{A.1}$$

and solve this equation together with equations (2.7a) and (2.7c):

$$xv' + xu' + v - w = \frac{x^2}{2}(u + 2v) \tag{A.2}$$

$$x^2v'' + 3xv' - xw' + v - w = \frac{x^2}{2}(w + 2v) \tag{A.3}$$

First Stage: the First Integrals of equations (A.1), (A.2) *and* (A.3)

The following intermediate step is helpful; let

$$p = u', \qquad q = v'$$

and we shall solve for p and q. First solve for w from (A.2) and differentiate to find w'

$$w = v(1 - x^2) - \frac{ux^2}{2} + x(p + q) \tag{A.4}$$

$$w' = q(2 - x^2) + p\left(1 - \frac{x^2}{2}\right) - 2xv - xu + x(p' + q') \tag{A.5}$$

Substituting w and w' in (A.3), we find

$$(x^2p)' = \tfrac{1}{2}x^3q + \tfrac{1}{2}x^2v(3 + x^2) + \tfrac{1}{4}x^2u(6 + x^2) \tag{A.6}$$

Consider now equation (A.1) and substitute for w and w'; we obtain

$$(x^2p)' - \frac{x^3q}{2} + \frac{x^3p}{2} + \frac{x^2v}{2}(3 - x^2) + \frac{x^2u}{4}(6 - x^2) = 0 \tag{A.7}$$

Equations (A.6) and (A.7) imply

$$p = 2xv + xu \tag{A.8}$$

already a first integral of our original equations (since $p = u'$). From (A.8) we can compute $(x^2p)'$

$$(x^2p)' = 3x^2(2v + u) + x^3(2q + p) \tag{A.9}$$

Using this result in equation (A.6) we infer

$$2v(q - x^2) + u(6 - x^2) + 6xq + 4xp = 0 \tag{A.10}$$

Equation (A.9) minus (A.7) implies

$$2v(q + x^2) + u(6 + x^2) + 6qx + 2px = 0 \qquad \text{(A.11)}$$

From this last pair of equations (twice (A.11) minus (A.10)) we infer

$$-xq = v(3 + x^2) + u(1 + x^2/2) \qquad \text{(A.12)}$$

This equation is also a first integral of our original equations. (Compare with the other first integral, namely equation (A.8).)

Second Stage: the Complete Solution

Having found p and q we proceed to find u, v and w.

Consider equations (A.4) and substitute for p and q from (A.8) and (A.12)

$$w = -2v - u \qquad \text{(A.13)}$$

However, notice that the first integrals (A.8) and (A.12) may be written as

$$v = u'/2x - \tfrac{1}{2}u \qquad \text{(A.14)}$$

$$-xv' = v(3 + x^2) + u(1 + x^2/2) \qquad \text{(A.15)}$$

Differentiating (A.14) we obtain

$$v' = u''/2x - u'(1 + x^2)/2x^2 \qquad \text{(A.16)}$$

We can now infer a second-order differential equation in u, by replacing v and v' in (A.15) from equations (A.14) and (A.16),

$$-\tfrac{1}{2}u'' - \frac{1}{2x}u' + \tfrac{1}{2}u = 0$$

whose solution is

$$u = a \exp(-x)/x \qquad \text{(A.17)}$$

Next, we may infer from (A.14) that

$$v = -\frac{a}{2x}\exp(-x)\left\{1 + \frac{1}{x} + \frac{1}{x^2}\right\} \qquad \text{(A.18)}$$

Finally, from (A.13) we infer that

$$w = \frac{a}{x^2}\exp(-x)\left(1 + \frac{1}{x}\right) \qquad \text{(A.19)}$$

Equations (A.17), (A.18) and (A.19) from the complete solution we required.

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